CH-6 Numerical Integration

- Numerical integration is also known as 'numerical quadrature' or 'mechanical quadrature' or 'approximate evaluation of integrals'.
- To integrate numerically $I(f) = \int_a^b f(x)\,dx$ we have to find the approximating polynomial $p(x)$ such that $f(x) \approx p(x)$ $\quad [f(x) \in C[a,b]$

So $I(p) = \int_a^b p(x)\,dx$

Now A/C to Weierstrass approximation theorem we know that for a given +ve number ε ∃ a polynomial $p(x)$ such that

$$|f(x) - p(x)| < \varepsilon / b-a \quad , \quad x \in [a,b]$$

ε is the error tolerance

then $|I(f) - I(p)| = \left|\int_a^b (f(x) - p(x))\,dx\right|$

$$\leq \int_a^b |f(x) - p(x)|\,dx$$

$$< \varepsilon$$

- There is a method to evaluate numerical quadrature.

(1) Newton-Cotes Method

(2) Gauss method.

NOTE

Riemann sum

$S = a_1 f(x_1) + a_2 f(x_2) + \cdots + a_n f(x_n)$

is a weighted sum and as can approximate to $I(f)$.

Also we can write as

$$R_n(f) = a_1 f(x_1) + a_2 f(x_2) + \cdots + a_n f(x_n)$$

and $R_n(f) \approx I(f)$

where $R_n(f)$ is called n-point integration rule. $x_1, x_2, \ldots x_n$ are called nodes or abscissas and $a_1, a_2, \ldots a_n$ are called the weights or coefficients.

now $E(f) = I(f) - R_n(f)$

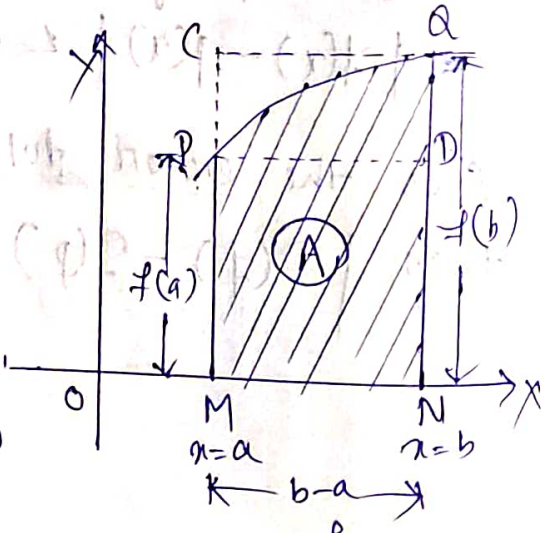
is called truncation error associated with the approximation $R_n(f)$ to $I(f)$.

Some simple quadrature Rule

1) Rectangular rule:-

Let $I(f) = \int_a^b f(x)\,dx$

- The geometrical meaning of $I(f)$ is the area A, bounded by curve $y = f(x)$, $x = a$, $x = b$.



- When the point $N \to M$, the area of MNPP is approximately equal to area A.

Similarly when $M \to N$, the area of MNQC is nearly equal to area A.

So $\int_a^b f(x)\,dx = (b-a) f(a)$

or $\int_a^b f(x)\,dx = (b-a) f(b)$

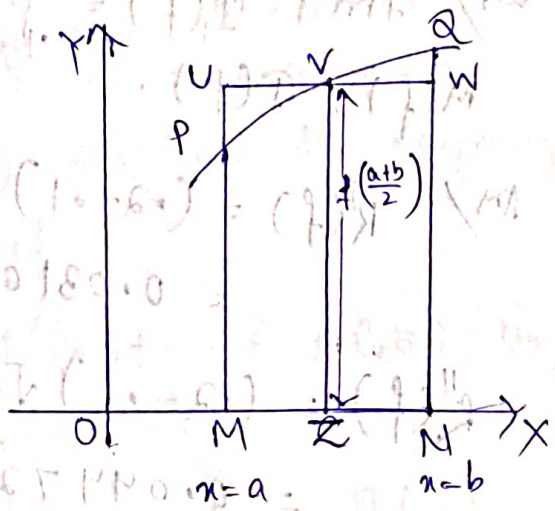
$$\Rightarrow \boxed{\begin{aligned} R'(f) &= (b-a)\, f(a) \\ R''(f) &= (b-a)\, f(b) \end{aligned}} \quad \text{---(1)}$$

which are respectively called as left rectangular rule and right rectangular rule.

(2) Mid point Rule :-

Here Z is the midpoint of MN.

So area of MNWV is approximately equal to area A.



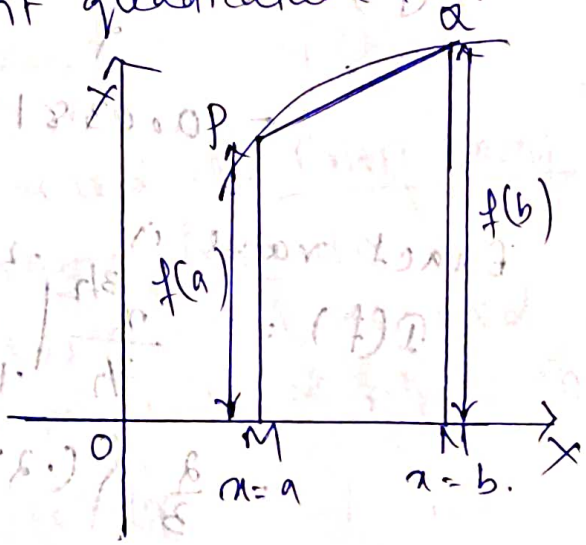
i.e $\int_a^b f(a)\, dx = (b-a)\, f\left(\frac{a+b}{2}\right)$

$$\Rightarrow \boxed{M(f) = (b-a)\, f\left(\frac{a+b}{2}\right)} \quad \text{---(2)}$$

This is called midpoint quadrature rule.

3) Trapizoidal rule :-

Here the Area of the trapezium MNPQ is approximately equal to area A.



So Area of trapizium

$$= \frac{1}{2} MN\,(MP + NQ)$$

$$= \frac{1}{2}(b-a)\left(f(a) + f(b)\right)$$

Thus $\boxed{T(f) = \frac{1}{2}(b-a)\{f(a) + f(b)\}}$ ---(3)

is called Trapizoidal rule.

NOTE

$$T(f) = \frac{1}{2}\left[R'(f) + R''(f)\right]$$

i.e $T(f)$ is the mean of left and right rectangular rule.

ex/ find $I_1(f) = \int_{.1}^{.2} \sqrt{x}\, dx$ using $R'(f), R''(f)$, $M(f)$, $T(f)$.

Sol/ $R'(f) = (.2-.1)\sqrt{.1}$

$= 0.03162278$

$R''(f) = (.2-.1)\sqrt{.2}$

$= 0.04472136$

$M(f) = (.2-.1)\sqrt{\frac{.1+.2}{2}}$

$= 0.03872983$

$T(f) = \frac{(.2-.1)}{2}\left(\sqrt{.1} + \sqrt{.2}\right)$

$= 0.03817207$

Exact value is

$$I(f) = \frac{x^{3/2}}{3/2}\Big|_{.1}^{.2} = \frac{2}{3}x^{3/2}\Big|_{.1}^{.2}$$

$$= \frac{2}{3}\left[(.2)^{3/2} - (.1)^{3/2}\right]$$

$= 0.0385466$

Name	$R'(f)$	$R''(f)$	$M(f)$	$T(f)$	Exact
$I(f)$	.03162278	.04472136	.03872983	.03817207	.0385466

Here $M(f)$ is more accurate.